How to Cope with the Curse of Dimensionality ?

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Curse of Dimensionality

ε	**error demand**
d	**the number of variables**
$n(\varepsilon, d)$	**the minimal cost**

Many problems suffer from the curse of dimensionality

$$n(\varepsilon, d) \ \geq c \ (1 + C)^{\,d}$$

for all $d = 1, 2, \ldots$ with $c > 0$ and $C > 0$.

IBC

IBC = Information-Based Complexity

- IBC is the branch of computational complexity that studies continuous mathematical problems.
- Typically, such problems are defined on spaces of functions of d variables. Often d is large.
- Typically, the available information is given by finitely many function values. Therefore it is partial, costly and often noisy.

Multivariate Integration for Korobov Spaces

$$r = \{r_j\} \quad \textbf{with} \quad 1 \le r_1 \le r_2 \le \cdots$$

H_{r_j}: 1-**periodic** $f : [0,1] \to \mathbb{C}$, $f^{(r_j - 1)}$ **abs. cont,** $f^{(r_j)} \in L_2$

$$\|f\|_{H_{r_j}}^2 = \left| \int_0^1 f(t)\,\mathrm{d}t \right|^2 + \int_0^1 \left| f^{(r_j)}(t) \right|^2 \mathrm{d}t$$

For $d \ge 1$**,**

$$H_{d,r} = H_{r_1} \otimes H_{r_2} \otimes \cdots \otimes H_{r_d}$$

Usually, it is assumed that $r_j \equiv r$

Multivariate Integration

For $f \in H_{d,r}$ we want to approximate

$$I_d(f) := \int_{[0,1]^d} f(t)\,\mathrm{d}t \qquad \approx \quad A_n(f)$$

- **Algorithms:**

$$A_n(f) = \phi_n(f(x_1), f(x_2), \dots, f(x_n)) \quad \text{with} \quad x_j \in [0,1]^d$$

- **Minimal Worst Case Error:**

$$e(n,d) = \inf_{A_n} \sup_{\|f\|_{H_{d,r}} \le 1} |I_d(f) - A_n(f)|$$

- **Information Worst Case Complexity:**

$$n(\varepsilon, d) \;=\; \min\{\, n \,|\; e(n,d) \le \varepsilon \,\}$$

Theorem

Let $r_j \equiv r$. Then there exists $c_r > 0$ and $C_r > 0$ such that

$$n(\varepsilon, d) \ > \ c_r \, (1 + C_r)^{\,d}$$

Based on Hickernell+W [2001] and Novak+W[2001], see also Sloan+W[2001]

Multivariate integration for Korobov space
with arbitrarily smooth functions
suffers from the curse of dimensionality

How to cope with the curse of dimensionality

- **switch to spaces with increased smoothness with respect to successive variables**

- **switch to weighted spaces, i.e., groups of variables are of varying importance**

- **switch to a more lenient setting, i.e, from the worst case setting to the randomized or average case setting**

Increasing Smoothness

Still the worst case setting and unweighted spaces with $r_1 \le r_2 \le \cdots$.

But we now allow to increase r_j

Let

$$R := \limsup_{k\to\infty} \frac{\ln\, k}{r_k}$$

Theorem

If $R < 2\,\ln\, 2\pi$ **then**

- **no curse**
- $n(\varepsilon,d) \le C\,\varepsilon^{-p(1+p/2)}$ **with** $p := \max(r_1^{-1}, R/\ln\, 2\pi) < 2,$
  i.e., strong polynomial tractability

Based on Papageorgiou+W [09], Kuo, Wasilkowski+W[09]

Weighted Spaces

Major research activities in last 20 years...

In particular, for $r_j \equiv r$ and $\gamma = \{\gamma_j\}$, redefine H_{r_j,γ_j} with

$$\|f\|^2_{H_{r_j,\gamma_j}} = \left|\int_0^1 f(t)\,\mathrm{d}t\right|^2 + \frac{1}{\gamma_j}\int_0^1 \left|f^{(r_j)}(t)\right|^2 \mathrm{d}t$$

For $d \geq 1$,

$$H_{d,r} = H_{r_1,\gamma_1} \otimes H_{r_2,\gamma_2} \otimes \cdots \otimes H_{r_d,\gamma_d}$$

Theorem

- **Gnewuch+W[08]**

  $\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{d} = 0$ **iff no curse,**

- **Hickernell+W[01]**

  $\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{\ln\, d} < \infty$ **iff polynomial tractability,**

  i.e., $n(\varepsilon, d) \le C\, d^q\, \varepsilon^{-p}$

- **Hickernell+W[01]**

  $\sum_{j=1}^{\infty} \gamma_j < \infty$ **iff strong polynomial tractability,**

  i.e., $n(\varepsilon, d) \le C\, \varepsilon^{-p}$

More Lenient Settings

From Worst Case Setting to

- **Randomized Setting**
- **Average Case Setting**

Average Case Setting $\leq$ Randomized Setting

Randomized Setting

- **Algorithms:**

$$A_{n,\omega}(f) = \phi_{n,\omega}(f(x_{1,\omega}), f(x_{2,\omega}), \dots, f(x_{n(\omega),\omega})) \quad \text{for a random } \omega$$

- **Minimal Randomized Error:**

$$e(n,d) = \inf_{A_n} \sup_{\|f\|_{H_{d,r}} \le 1} \left[\mathbb{E}\, |I_d(f) - A_{n,\omega}(f)|^2\right]^{1/2}$$

- **Information Randomized Complexity:**

$$n(\varepsilon, d) \;=\; \min\{\, n \,|\;\; e(n,d) \le \varepsilon \,\}$$

Monte Carlo Algorithm

$$A_{n,\omega}(f) = \frac{1}{n}\sum_{j=1}^{n} f(x_{j,\omega})$$

with

$x_{j,\omega}$ **iid with uniform distribution over** $[0,1]^d$

- $n(\varepsilon, d) \le \varepsilon^{-2}$
- **no curse and strong polynomial tractability**

Conclusions

- Many multivariate problems suffer from the curse of dimensionality in the worst case setting
- We may sometimes break the curse of dimensionality by
  - switching to spaces with increased smoothness with respect to successive variables
  - switching to weighted spaces, i.e., groups of variables are of varying importance
  - switching to a more lenient setting, i.e, from the worst case setting to the randomized or average case setting

Book

More can be found in

Tractability of Multivariate Problems

Erich Novak and Henryk Woźniakowski

- **Volume I: Linear Information (2008)**
- **Volume II: Standard Information for Functionals (2010)**
- **Volume III: Standard Information for Operators (2012)**