Dynamical bifurcations and singularly perturbed systems of differential equations

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Part I: Singular Perturbations

Motivating example – a dengue fever model.

Assumptions:

a) Host population: susceptible S_h, infectives I_h, recovered with immunity R_h, Malthusian demography,

b) Vector population: susceptible S_v, infective I_v, balanced population: $S_v(t) + I_v(t) = M_0$,

c) Vector population smaller that the host population,

d) Non-lethal.

Then

$$
\begin{aligned}
S_h' &= (\Psi_h - \mu_{1h})S_h + \Psi_h I_h + (\Psi_h + \rho_h)R_h - \sigma_v \beta_{hv} \frac{I_v S_h}{N_h}, \\
I_h' &= \sigma_v \beta_{hv} \frac{I_v S_h}{N_h} - (\gamma_h + \mu_{1h}) I_h, \\
R_h' &= \gamma_h I_h - (\rho_h + \mu_{1h}) R_h, \\
S_v' &= \mu_{1v} S_v - \sigma_v \beta_{vh} \frac{I_h S_v}{N_h}, \\
I_v' &= -\mu_{1v} I_v + \sigma_v \beta_{vh} \frac{I_h S_v}{N_h}.
\end{aligned} \tag{1}
$$

Table: Parameter values

Parameters	day^{-1}	year^{-1}
Ψ_h	7.666×10^{-5}	2.8×10^{-2}
γ_h	3.704×10^{-3}	1.352×10^{0}
δ_h	3.454×10^{-4}	1.261×10^{-1}
ρ_h	1.460×10^{-2}	5.33×10^{0}
μ_{1h}	4.212×10^{-5}	1.5×10^{-2}
σ_v	0.6	2.19×10^{2}
μ_{1v}	0.1429	5.2×10^{1}
	Dimensionless parameters	
β_{vh}	0.8333	
β_{hv}	2×10^{-2}	

Problem: The original models are too complex for a robust analysis and may yield redundant information for particular applications.

Aim: to build a simpler (macro) model for the evolution of macro-variables relevant for a chosen time scale which, for these variables, retains the main features of the dynamics of the detailed (micro) model. The process often is referred to as the aggregation, or lumping, of states.

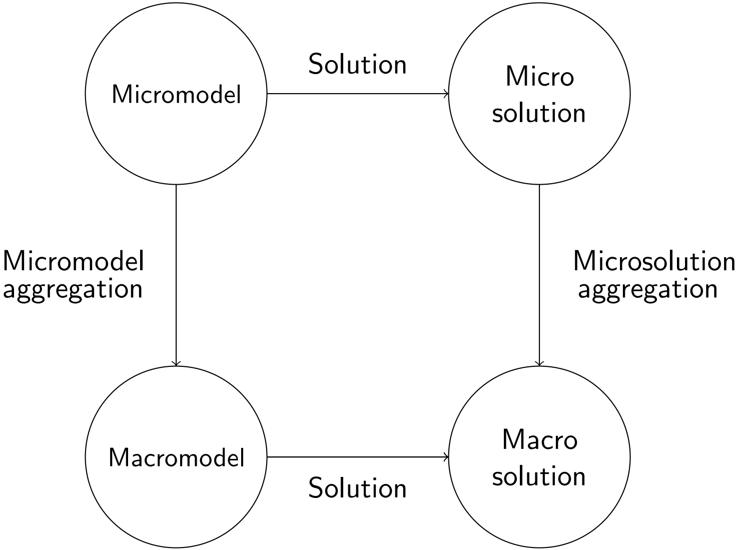


Figure: Commutativity of the aggregation diagram

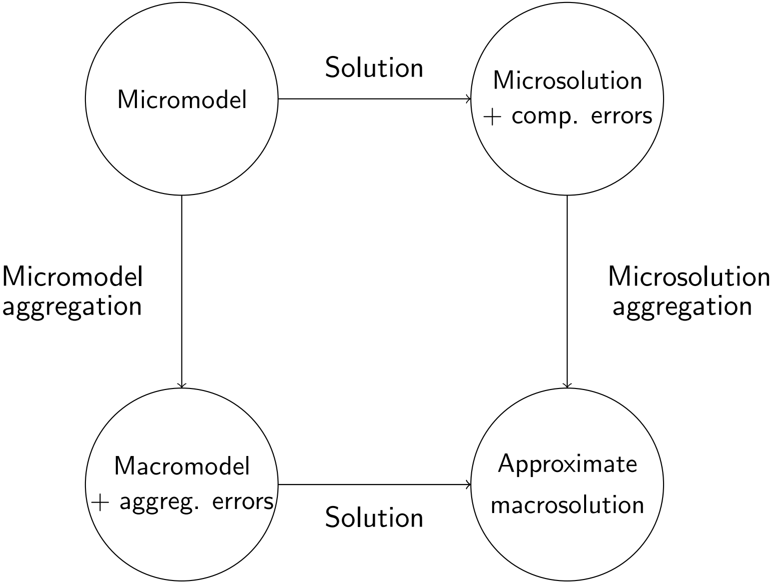


Figure: Approximate commutativity of the aggregation diagram

Tikhonov theorem — aggregation in systems of ODEs

We are concerned with models in which the existence of two characteristic time scales leads to singularly perturbed systems

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(t, \mathbf{x}, \mathbf{y}, \epsilon), \qquad \mathbf{x}(0) = \mathring{\mathbf{x}}, \\ \epsilon \mathbf{y}' &= \mathbf{g}(t, \mathbf{x}, \mathbf{y}, \epsilon), \qquad \mathbf{y}(0) = \mathring{\mathbf{y}}, \end{aligned} \tag{2}$$

where $'$ denotes differentiation with respect to t and $\mathbf{f}$ and $\mathbf{g}$ are sufficiently regular functions from open subsets of $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+$ to, respectively, $\mathbb{R}^n$ and $\mathbb{R}^m$, for some $n, m \in \mathbb{N}$.

Tikhonov theorem gives conditions ensuring that the solutions $(\mathbf{x}_\epsilon(t), \mathbf{y}_\epsilon(t))$ of (2) converge to $(\bar{\mathbf{x}}(t), \bar{\mathbf{y}}(t, \bar{\mathbf{x}}))$, where $\bar{\mathbf{y}}(t, \mathbf{x})$ is the solution to the equation

$$0 = \mathbf{g}(t, \mathbf{x}, \mathbf{y}, 0), \tag{3}$$

called the *quasi steady state*, and $\bar{\mathbf{x}}(t)$ is the solution of

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}, \bar{\mathbf{y}}(t, \mathbf{x}), 0), \qquad \mathbf{x}(0) = \overset{\circ}{\mathbf{x}}, \tag{4}$$

obtained from the first equation of (2) by substituting the unknown $\mathbf{y}$ by the known quasi steady state $\bar{\mathbf{y}}$.

Main assumptions:

- the quasi-steady states are isolated in some set $[0, T] \times \bar{\mathcal{U}}$;
- for each fixed t and $\mathbf{x}$, the quasi steady state solution $\bar{\mathbf{y}}(t, \mathbf{x})$ of (3) is an asymptotically stable equilibrium of

$$\frac{d\,\tilde{\mathbf{y}}}{d\,\tau} = \mathbf{g}(t, \mathbf{x}, \tilde{\mathbf{y}}, 0); \qquad (5)$$

- $\bar{\mathbf{x}}(t) \in \mathcal{U}$ for $t \in [0, T]$ provided $\overset{\circ}{\mathbf{x}} \in \bar{\mathcal{U}}$;
- $\overset{\circ}{\mathbf{y}}$ belongs to the basin of attraction of $\bar{\mathbf{y}}(0, \overset{\circ}{\mathbf{x}})$.

Then the following theorem is true.

Theorem

Let the above assumptions be satisfied. Then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in]\,0, \varepsilon_0]$ *there exists a unique solution* $(\mathbf{x}_\varepsilon(t), \mathbf{y}_\varepsilon(t))$ *of Problem (2) on* $[0, T]$ *and*

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbf{x}_\varepsilon(t) &= \bar{\mathbf{x}}(t), \qquad t \in [0, T]\,, \\ \lim_{\varepsilon\to 0} \mathbf{y}_\varepsilon(t) &= \bar{\mathbf{y}}(t), \qquad t \in]\,0, T]\,, \end{aligned} \tag{6}$$

where $\bar{\mathbf{x}}(t)$ *is the solution of (4) and* $\bar{\mathbf{y}}(t) = \bar{\mathbf{y}}(t, \bar{\mathbf{x}}(t))$ *is the solution of (3).*

Back to the model.

With $\epsilon = \frac{1}{1000}$, (1) can be written as

$$
\begin{aligned}
S_h' &= 0.013S_h + 0.028I_h + 5.358R_h - 4.38\frac{I_v S_h}{N_h},\\
I_h' &= 4.38\frac{I_v S_h}{N_h} - 1.367I_h,\\
R_h' &= 1.352I_h - 5.345R_h,\\
\epsilon S_v' &= 0.052S_v - 0.182\frac{I_h S_v}{N_h},\\
\epsilon I_v' &= -0.052I_v + 0.182\frac{I_h S_v}{N_h}.
\end{aligned}
\tag{7}
$$

The Tikhonov theorem allows for the reduction of (1) to a SIR system

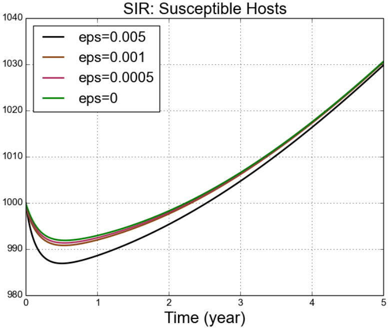
$$\begin{aligned}\bar{S}_h' &= (\Psi_h - \mu_{1h})\bar{S}_h + \Psi_h \bar{I}_h + (\Psi_h + \rho_h)\bar{R}_h - \lambda(t)\bar{S}_h,\\ \bar{I}_h' &= \lambda(t)\bar{S}_h - (\gamma_h + \mu_{1h})\bar{I}_h,\\ \bar{R}_h' &= \gamma_h \bar{I}_h - (\rho_h + \mu_{1h})\bar{R}_h,\end{aligned} \tag{8}$$

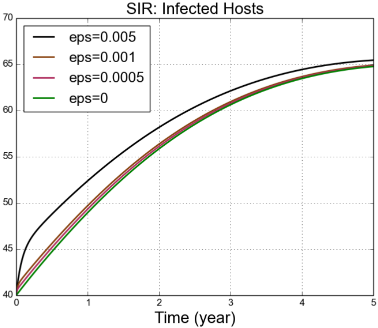
with modified infection force

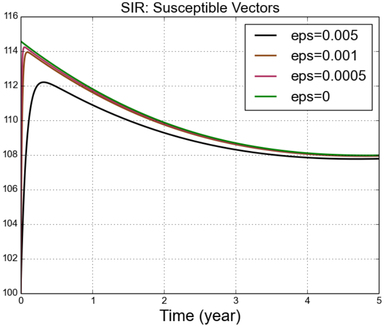
$$\lambda = \frac{\sigma_v \beta_{hv}}{\bar{N}_h}\bar{I}_v = \frac{\sigma_v \beta_{hv}}{\bar{N}_h}\frac{\sigma_v \beta_{vh} M_0 \bar{I}_h}{\mu_{1v}\bar{N}_h + \sigma_v \beta_{vh}\bar{I}_h}, \tag{9}$$

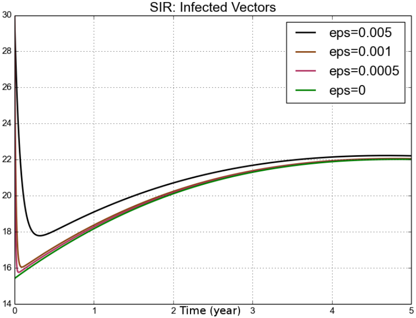
where

$$\bar{N}_h(t) = N_h(0)e^{(\Psi_h - \mu_{1h})t}.$$


SIR: Susceptible Hosts
eps=0.005
eps=0.001
eps=0.0005
eps=0
1040
1030
1020
1010
1000
990
980
0
1
2
3
4
5
Time (year)


SIR: Infected Hosts
eps=0.005
eps=0.001
eps=0.0005
eps=0
70
65
60
55
50
45
40
0
1
2
3
4
5
Time (year)


SIR: Susceptible Vectors
eps=0.005
eps=0.001
eps=0.0005
eps=0
Time (year)


SIR: Infected Vectors
eps=0.005
eps=0.001
eps=0.0005
eps=0
Time (year)

Part II: Dynamic bifurcations

In the classical bifurcation theory we consider the differential equation

$$\dot{y} = g(x, y), \tag{10}$$

where x is a parameter, and investigate the character of the equilibrium $y^* = y^*(x)$; that is, the solution to

$$0 = g(x, y),$$

when x passes through some exceptional values, called the bifurcation points.

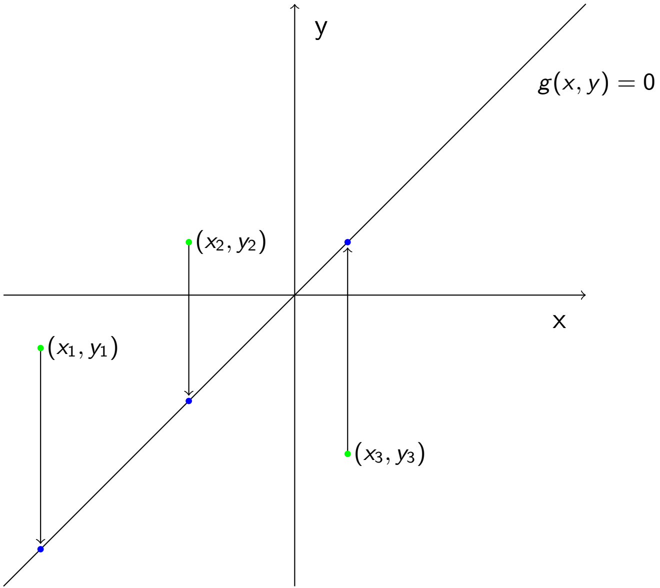


Figure: Dynamics described by Eqn (14) if $g(x, y) = 0$ is attractive.

If we move x according to some rule $\tau \to x(\tau)$, then modified (14):

$$\dot{y}(\tau) = g(x(\tau), y(\tau)), \tag{11}$$

will generate a 'long-term' dynamics on the manifold $g(x, y) = 0$.

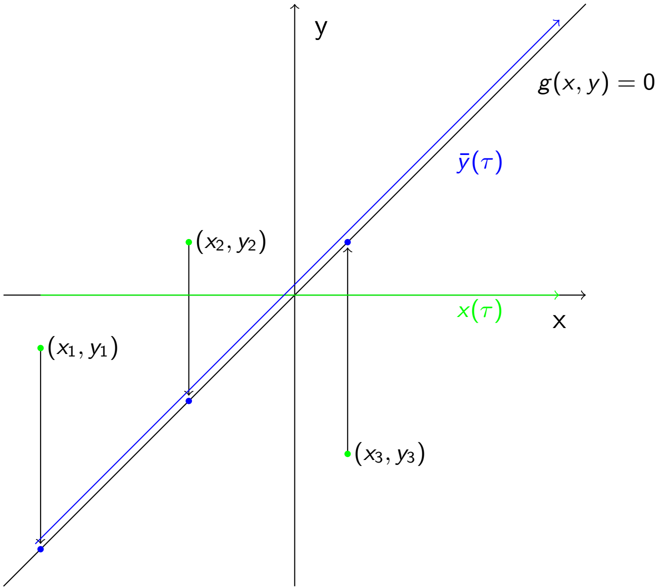


Figure: Moving x generates a dynamics on the manifold $g(x, y) = 0$.

In general, the bifurcation parameter can be coupled with the main equation:

$$\begin{aligned} \dot{\mathbf{x}} &= \epsilon \mathbf{f}(\mathbf{x}, \mathbf{y}), \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}). \end{aligned} \tag{12}$$

Changing time as $\epsilon\tau = t$ we obtain (2),

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(\mathbf{x}, \mathbf{y}), \\ \epsilon \mathbf{y}' &= \mathbf{g}(\mathbf{x}, \mathbf{y}); \end{aligned} \tag{13}$$

that is, a singularly perturbed system in the Tikhonov form.

Hence, long term dynamics of (12) is equivalent to small ϵ dynamics of (2). Both problems are equivalent for $\epsilon > 0$. On the other hand, we may ask how well the solutions of (14) and (15) with $\epsilon = 0$:

$$\begin{aligned} \dot{\mathbf{x}} &= 0, \\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{14}$$

(fast dynamics) and

$$\begin{aligned} \mathbf{x}' &= \mathbf{f}(\mathbf{x}, \mathbf{y}), \\ 0 &= \mathbf{g}(\mathbf{x}, \mathbf{y}), \end{aligned} \tag{15}$$

(slow dynamics) approximate the true solution?

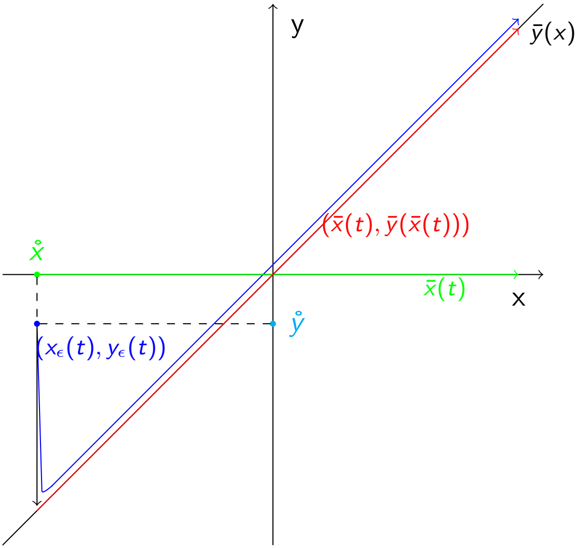


Figure: Dynamics described by Eqns (12) and (15) by the Tikhonov theorem.

Quite often, however, $g(x, y) = 0$ has branching solutions.

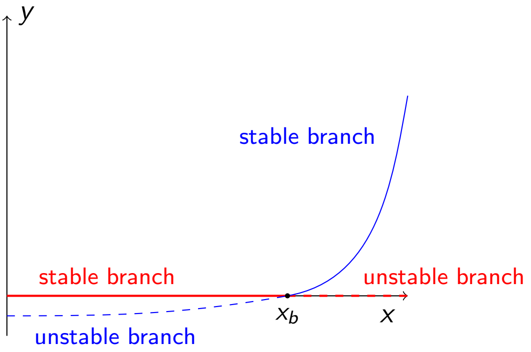


Figure: Transcritical bifurcation at the bifurcation point x_b

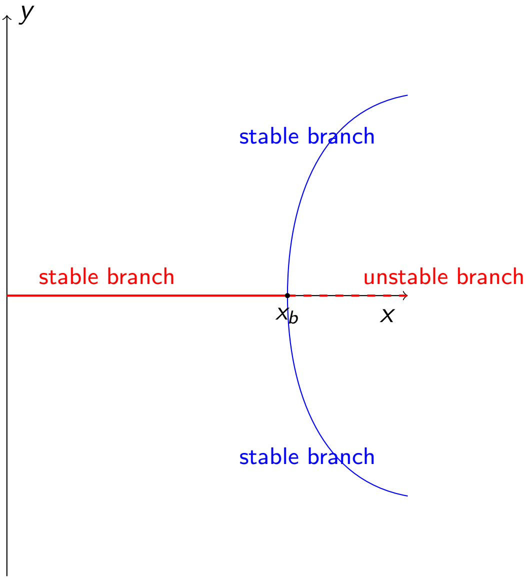


Figure: Hopf bifurcation at the bifurcation point x_b

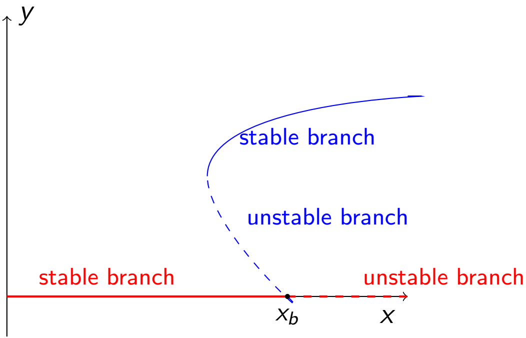


Figure: Backward bifurcation at the bifurcation point x_b

By classical bifurcation theory, it is expected that the solutions to (12) should converge to the equilibria $y^*(x)$ of

$$y' = g(x, y)$$

whenever they are attracting. In terms of (15), the solutions $y_\epsilon(t)$ should converge to the quasi steady states; that is, to solutions to

$$g(x, y) = 0,$$

whenever they are attracting.

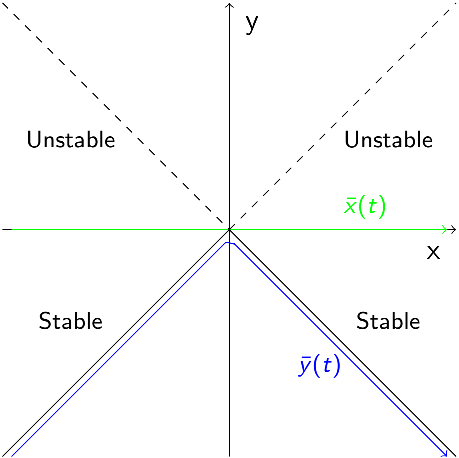


Figure: Expected behaviour of solutions in the case of transcritical bifurcation.

Delayed asymptotic switch.

An SIS model with demography for a quick disease often can be reduced to

$$\begin{aligned}\epsilon i_\epsilon' &= -\epsilon\mu i_\epsilon + (\lambda i_\epsilon(n - i_\epsilon) - \gamma i_\epsilon),\\ i_\epsilon(0) &= i_0,\end{aligned} \tag{16}$$

where i_ϵ is the density of infectives, μ is the death rate in the population, λ is the force of infection, γ is the recovery rate and

$$n(t) = n_0 e^{rt},$$

where $r > 0$ is the net growth rate in the population.

The quasistationary states of (16) are

$$\phi_1 = 0, \qquad \phi_2 = n_0 e^{rt} - \nu,$$

where $\nu = \gamma/\lambda$. They intersect at

$$t_c = \frac{1}{r} \log\left(\frac{\nu}{n_0}\right) > 0. \tag{17}$$

provided $n_0 < \nu$.

Then

- ϕ_1 is attractive for $0 < t < t_c$ i repelling for $t > t_c$;
- ϕ_2 is attractive for $t > t_c$ and negative (hence irrelevant) for $0 < t < t_c$).

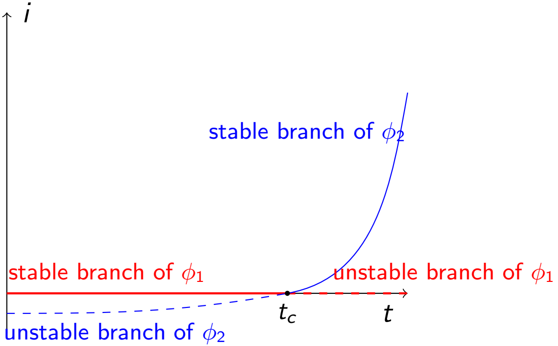


Figure: Geometry of the quasisteady states.

Eqn (16) is the Bernoulli equation whose solution is

$$i_\epsilon(t) = \frac{i_0 e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{1 + \frac{\lambda i_0}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s} ds}, \tag{18}$$

where

$$G(t,\epsilon) = \frac{n_0\lambda}{r}(e^{rt} - 1) - \gamma t - \epsilon\mu t.$$

The limit of i_ϵ as $\epsilon \to 0$ depends on the sign of $G(t,0)$.

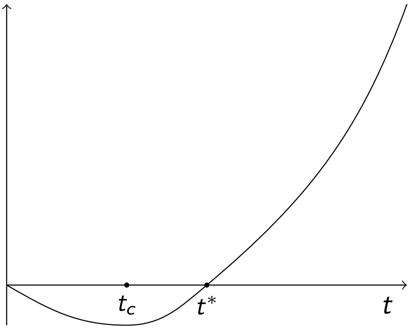


Figure: The shape of $G(t,0)$

We have

- $G(t,0) < 0$ dla $0 < t < t^*$;
- $G(t,0) > 0$ dla $t < t^*$.

Then

$$
\begin{aligned}
0 \;\leq\; \lim_{\epsilon\to 0} i_\epsilon(t) &= \lim_{\epsilon\to 0} \frac{i_0 e^{\frac{1}{\epsilon}G(t,0)-\mu t}}{1+\frac{\lambda i_0}{\epsilon}\int_0^t e^{\frac{1}{\epsilon}G(s,0)-\mu s}ds} \\
\leq\; & \lim_{\epsilon\to 0} i_0 e^{\frac{1}{\epsilon}G(t,0)-\mu t} = 0 = \phi_1
\end{aligned}
$$

for$t \in (0, t^*)$, hence $i_\epsilon(t)$ is close to $\phi_1 = 0$ also for $t \in]t_c, t^*[$,

when ϕ_1 already is repelling.

Contrary to naive numerical simulations, t^* does not depend on ϵ.

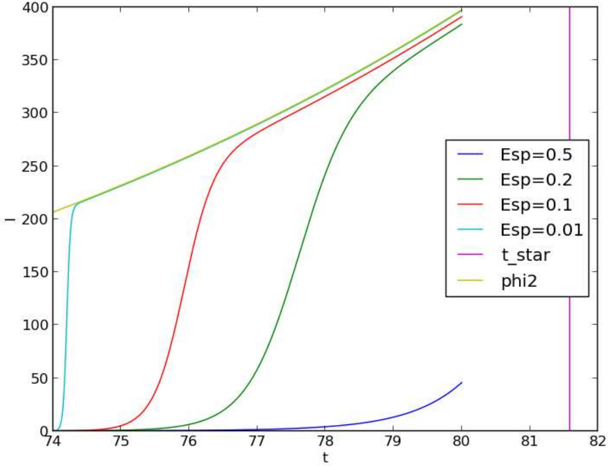


Figure: Solutions for (16) using standard ODE solver in Python.

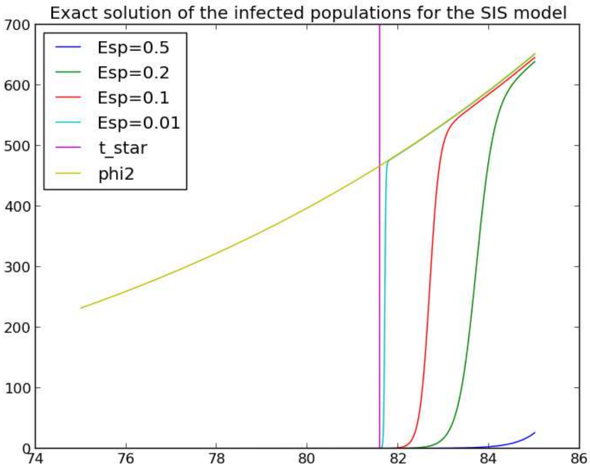


Figure: Solutions for (16) using (18).

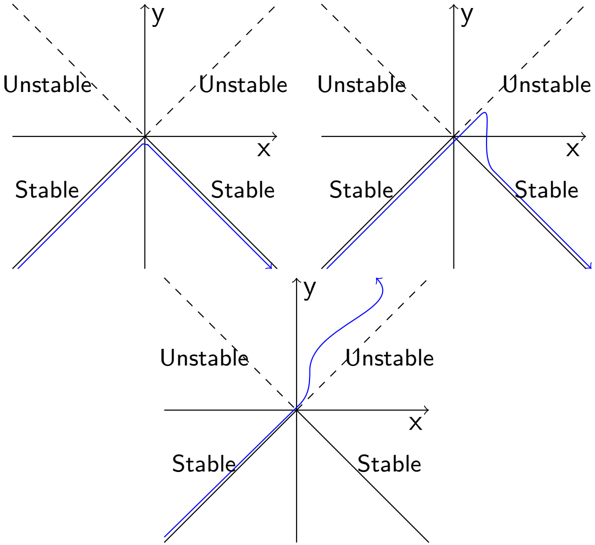


Figure: Possible behaviour of solutions in the case of transcritical bifurcation.

Part III: Some mathematics

1-dimensional theory – the method of upper and lower solutions (Butuzov 2004)

Let us consider a singularly perturbed scalar differential equation.

$$\begin{aligned} \epsilon\frac{dy}{dt} &= g(y,t,\epsilon), \quad t\in(0,T) \\ y(0,\epsilon) &= \mathring{y}, \end{aligned} \tag{19}$$

Define

$$G(t,\epsilon)=\int_0^t g_y(0,s,\epsilon)ds.$$

(A_1) $g(y,t,0)=0$ has two roots $y=\phi_1(t)\equiv 0$ and $y=\phi_2(t)$, which intersect at $t=t_c\in(0,T)$. We assume that

$$\phi_2(t)<0 \text{ for } 0\le t\le t_c,\quad \phi_2(t)>0 \text{ for } t_c\le t\le T.$$

(A_2) Stability of the quasi steady states: $\phi_1(t)=0$ is attractive on $[0,t_c)$ and repelling on $(t_c,T]$ and $\phi_2(t)$ is attractive on $(t_c,T]$;

(A_3) $g(0,t,\epsilon)\equiv 0$ for $(t,\epsilon)\in\bar{U}_T\times\bar{I}_{\epsilon_0}$.

(A_4) The equation $G(t,0)=0$ has a root $t^*\in(t_0,T)$.

(A_5) There is a positive number c_0 such that $\pm c_0\in I_y$ and

$$g(y,t,\epsilon)\le g_y(0,t,\epsilon)y \text{ for } t\in[0,t^*],\ |y|\le c_0.$$

Theorem

Let us assume that all assumptions $(A_1) - (A_5)$ hold. If $\mathring{y} > 0$ then for sufficiently small ϵ there exists a unique solution $y(t, \epsilon)$ of (19) that is positive and

$$\lim_{\epsilon \to 0} y(t, \epsilon) = 0 \text{ for } t \in (t_0, t^*), \tag{20}$$

$$\lim_{\epsilon \to 0} y(t, \epsilon) = \phi_2(t) \text{ for } t \in (t^*, T). \tag{21}$$

If $\mathring{y} < 0$, then the unique solution is negative, converges to $y = 0$ as $\epsilon \to 0$ on $(0, t^*)$ and escapes from the unstable root $y = 0$ for $t > t^*$.

The idea of the proof.

As long as $y(t,\epsilon),\epsilon$ are small, we can approximate

$$\epsilon y'(t,\epsilon) = g(y(t,\epsilon),t,\epsilon) \approx g_y(0,t,0)y(t,\epsilon) \qquad (22)$$

so that

$$y(t,\epsilon) \approx \mathring{y} e^{\frac{1}{\epsilon}\int\limits_0^t g_y(0,s,0)ds}.$$

We have $y(t,\epsilon) \to 0$ as $\epsilon \to 0$ if the exponent is negative. By assumption, $g_y < 0$ on $]0, t_c[$ but then the integral stays negative on a larger interval, hence $t^* > t_c$.

However, to make this work, (A5) is needed so that the solution of the linearized equation (22) is the upper solution.