How to Cope with the Curse of Dimensionality?

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Curse of Dimensionality

\[ n(\varepsilon, d) \geq c (1 + C)^d \]

for all \( d = 1, 2, \ldots \) with \( c > 0 \) and \( C > 0 \).

Many problems suffer from the curse of dimensionality.
IBC = Information-Based Complexity

- IBC is the branch of computational complexity that studies continuous mathematical problems.
- Typically, such problems are defined on spaces of functions of \( d \) variables. Often \( d \) is large.
- Typically, the available information is given by finitely many function values. Therefore it is partial, costly and often noisy.
Multivariate Integration for Korobov Spaces

\[ r = \{r_j\} \quad \text{with} \quad 1 \leq r_1 \leq r_2 \leq \cdots \]

\[ H_{r_j} : \quad \text{1-periodic } f : [0, 1] \to \mathbb{C}, \quad f^{(r_j-1)} \text{ abs. cont, } f^{(r_j)} \in L_2 \]

\[ \|f\|_{H_{r_j}}^2 = \left| \int_0^1 f(t) \, dt \right|^2 + \int_0^1 \left| f^{(r_j)}(t) \right|^2 \, dt \]

For \( d \geq 1 \),

\[ H_{d,r} = H_{r_1} \otimes H_{r_2} \otimes \cdots \otimes H_{r_d} \]

Usually, it is assumed that \( r_j \equiv r \)
Multivariate Integration

For $f \in H_{d,r}$ we want to approximate

$$I_d(f) := \int_{[0,1]^d} f(t) \, dt \approx A_n(f)$$

- **Algorithms:**
  $$A_n(f) = \phi_n(f(x_1), f(x_2), \ldots, f(x_n)) \text{ with } x_j \in [0,1]^d$$

- **Minimal Worst Case Error:**
  $$e(n, d) = \inf_{A_n} \sup_{\|f\|_{H_{d,r}} \leq 1} |I_d(f) - A_n(f)|$$

- **Information Worst Case Complexity:**
  $$n(\varepsilon, d) = \min \{ n \mid e(n, d) \leq \varepsilon \}$$
Theorem

Let \( r_j \equiv r \). Then there exists \( c_r > 0 \) and \( C_r > 0 \) such that

\[
n(\varepsilon, d) > c_r (1 + C_r)^d
\]

Based on Hickernell+W [2001] and Novak+W[2001], see also Sloan+W[2001]

Multivariate integration for Korobov space with arbitrarily smooth functions suffers from the curse of dimensionality
How to cope with the curse of dimensionality

- switch to spaces with increased smoothness with respect to successive variables

- switch to weighted spaces, i.e., groups of variables are of varying importance

- switch to a more lenient setting, i.e., from the worst case setting to the randomized or average case setting
Increasing Smoothness

Still the worst case setting and unweighted spaces with \( r_1 \leq r_2 \leq \cdots \).

But we now allow to increase \( r_j \)

Let

\[
R := \limsup_{k \to \infty} \frac{\ln k}{r_k}
\]

Theorem

If \( R < 2 \ln 2\pi \) then

- no curse
- \( n(\varepsilon, d) \leq C \varepsilon^{-p(1+p/2)} \) with \( p := \max(r_1^{-1}, R/\ln 2\pi) < 2 \), i.e., strong polynomial tractability

Based on Papageorgiou+W [09], Kuo, Wasilkowski+W[09]
Weighted Spaces

Major research activities in last 20 years...

In particular, for $r_j \equiv r$ and $\gamma = \{\gamma_j\}$, redefine $H_{r_j, \gamma_j}$ with

$$\|f\|_{H_{r_j, \gamma_j}}^2 = \left| \int_0^1 f(t) \, dt \right|^2 + \frac{1}{\gamma_j} \int_0^1 \left| f^{(r_j)}(t) \right|^2 \, dt$$

For $d \geq 1$,

$$H_{d,r} = H_{r_1, \gamma_1} \otimes H_{r_2, \gamma_2} \otimes \cdots \otimes H_{r_d, \gamma_d}$$
Theorem

- **Gnewuch+W[08]**
  \[ \lim_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j}{d} = 0 \quad \text{iff} \quad \text{no curse}, \]

- **Hickernell+W[01]**
  \[ \limsup_{d \to \infty} \frac{\sum_{j=1}^{d} \gamma_j}{\ln d} < \infty \quad \text{iff} \quad \text{polynomial tractability}, \]
  \[ \text{i.e.,} \quad n(\varepsilon, d) \leq C d^q \varepsilon^{-p} \]

- **Hickernell+W[01]**
  \[ \sum_{j=1}^{\infty} \gamma_j < \infty \quad \text{iff} \quad \text{strong polynomial tractability}, \]
  \[ \text{i.e.,} \quad n(\varepsilon, d) \leq C \varepsilon^{-p} \]
More Lenient Settings

From Worst Case Setting to

- Randomized Setting
- Average Case Setting

Average Case Setting $\leq$ Randomized Setting
Randomized Setting

- **Algorithms:**

  \[ A_{n,\omega}(f) = \phi_{n,\omega}(f(x_{1,\omega}), f(x_{2,\omega}), \ldots, f(x_{n(\omega),\omega})) \text{ for a random } \omega \]

- **Minimal Randomized Error:**

  \[
  e(n, d) = \inf_{A_n} \sup_{\|f\|_{H_d, r} \leq 1} \left[ \mathbb{E} \left| I_d(f) - A_{n,\omega}(f) \right|^2 \right]^{1/2}
  \]

- **Information Randomized Complexity:**

  \[
  n(\varepsilon, d) = \min \{ n \mid e(n, d) \leq \varepsilon \}
  \]
Monte Carlo Algorithm

\[ A_{n,\omega}(f) = \frac{1}{n} \sum_{j=1}^{n} f(x_{j,\omega}) \]

with

\[ x_{j,\omega} \text{ iid with uniform distribution over } [0, 1]^d \]

- \[ n(\varepsilon, d) \leq \varepsilon^{-2} \]
- no curse and strong polynomial tractability
Conclusions

- Many multivariate problems suffer from the curse of dimensionality in the worst case setting

- We may sometimes break the curse of dimensionality by
  - switching to spaces with increased smoothness with respect to successive variables
  - switching to weighted spaces, i.e., groups of variables are of varying importance
  - switching to a more lenient setting, i.e., from the worst case setting to the randomized or average case setting
More can be found in

**Tractability of Multivariate Problems**

Erich Novak and Henryk Woźniakowski

- **Volume I:** Linear Information (2008)
- **Volume II:** Standard Information for Functionals (2010)
- **Volume III:** Standard Information for Operators (2012)